TWO-FERMION COMPOSITE QUASI-BOSONS AND DEFORMED OSCILLATORS

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The concept of quasi-bosons or composite bosons (like mesons, excitons, *etc.*) has a wide range of potential physical applications. Even composed of two pure fermions, the quasi-boson creation and annihilation operators satisfy non-standard commutation relations. It is natural to try to realize the quasi-boson operators by the operators of a deformed (nonlinear) oscillator, the latter constituting a widely studied field of modern quantum physics. In this paper, it is proven that the deformed oscillators which realize quasi-boson operators in a consistent way really exist. The conditions for such realization are derived, and the uniqueness of the family of deformations under consideration is shown.

1. Introduction

The study of many-body problems that involve composite particles essentially differs from those for pointlike particles because of the necessity of a more complicated treatment. Namely, because of the internal degrees of freedom due to constituents, the statistical properties of the composite (thus, not pointlike) particles may essentially deviate from the purely Bose or Fermi description. Such deviation, encapsulated in modified commutation relations, can be appropriately modeled (represented) by adopting some deformed (say, q- or p,q-deformed or yet another) version of the oscillator algebra. A particular realization of the idea to describe composite bosons (or "quasi-bosons", see [1]) in terms of a deformed Heisenberg algebra was demonstrated by Avancini and Krein in [2] who utilized the quonic version [3] of the deformed boson algebra. It should be stressed that these quons differ from the widely explored (system of) deformed oscillators of the Arik–Coon type [4] if more than one mode are considered: in that case, all modes of the Arik–Coon type are independent (that means the mutual commuta-

tion of the operators corresponding to different modes), unlike the quons whose different modes do not commute, see [2, 3].

Although models of deformed oscillators are known in a diversity of versions [5–10], to the best of our knowledge, a detailed analysis of possible realizations, on their base, of quasi-bosons is lacking in the literature. The present paper can be considered as a step to fill this gap and contains some results in that direction. Namely, we carry out the detailed analysis in the important case dealing with a set of independent modes (copies) of deformed oscillators, whereas, for the individual copy, we examine the most general possible structure function $\phi(N)$ of deformation which, as is well known, unambiguously determines [11, 12] the deformed algebras, i.e. the form of basic commutation relations for the annihilation, creation, and number operators, according to the formula $aa^\dagger - a^\dagger a = \phi(N+1) - \phi(N)$.

Diverse models of deformed oscillators, due to their peculiar properties, have received much attention for the last two decades. Among the best known and extensively studied deformed oscillator models, there are such as the q-deformed Arik–Coon (AC) [4] and Biedenharn–Macfarlane (BM) ones [5], as well as the two-parameter p,q-deformed oscillator [8]. Besides these, there exists the q-deformed Tamm–Dancoff (TD) oscillator [6], also explored though to a much lesser extent [7]. Unlike all the mentioned models, there is a very modest knowledge concerning the so-called μ-deformed oscillator. Introduced in [10], the μ-oscillator shows essentially different features and exhibits rather unusual properties [13].

Being direct extensions of the standard quantum harmonic oscillator, the deformed oscillators find a diversity of applications in the description of miscellaneous physical systems involving essential nonlinearities, from, say,

quantum optics and the Landau problem to high-energy quantum particle phenomenology and modern quantum field theory (see, e.g., [14–21]. That is why the possible use, in order to realize quasi-bosons, of deformed oscillators (deformed bosons) is very desirable due to a considerable simplification of the relevant analysis, achieved when the algebra representing the initial system of composite particles reduces to the algebra corresponding to some deformed oscillator. In this sense, the information about the internal structure of particles is carried by one or more parameters of deformation. The present work realizes the just mentioned procedure of reducing for the case of quasi-bosons [2] consisting of two ideal fermions. The obtained results are of value since, for *composite physical particles or quasi-particles* (say, mesons, higgsons, light even nuclei, excitons, etc.), it would be very useful to have them realized as deformed bosons, using deformed oscillators.

2. Quasi-Bosons as Two-Fermion Composites

We consider, like in [2], the system of composite boson-like particles (or quasi-bosons, see [1]) such that each copy/mode of them is composed of two usual fermions. First of all, we will study the realization of quasi-bosons in terms of a set of independent identical copies of deformed AC oscillators [4].

By $a_\mu^\dagger$, $b_\nu^\dagger$, a_μ, b_ν, we denote, respectively, the creation and destruction operators of the two (mutually anticommuting) sets of usual fermions, with standard anticommutation relations, and use these fermions to construct quasi-bosons. Then, the quasi-bosonic creation and destruction operators $A_\alpha^\dagger$, A_α (where α labels a particular quasi-boson and denotes the whole set of its quantum numbers) are given as

$$A_\alpha^\dagger = \sum_{\mu\nu} \Phi_\alpha^{\mu\nu} a_\mu^\dagger b_\nu^\dagger, \quad A_\alpha = \sum_{\mu\nu} \overline{\Phi}_\alpha^{\mu\nu} b_\nu a_\mu, \tag{1}$$

where $a_\mu^\dagger$, $b_\nu^\dagger$, a_μ, b_ν obey the relations

$$\{a_\mu, a_{\mu'}^\dagger\} \equiv a_\mu a_{\mu'}^\dagger + a_{\mu'}^\dagger a_\mu = \delta_{\mu\mu'}, \quad \{a_\mu, a_{\mu'}\} = 0,$$
$$\{b_\nu, b_{\nu'}^\dagger\} \equiv b_\nu b_{\nu'}^\dagger + b_{\nu'}^\dagger b_\nu = \delta_{\nu\nu'}, \quad \{b_\nu, b_{\nu'}\} = 0$$

and, in addition, each of $a_\mu^\dagger$, a_μ anticommutes with each of $b_\nu^\dagger$, b_ν. One can easily check that

$$[A_\alpha, A_\beta] = [A_\alpha^\dagger, A_\beta^\dagger] = 0. \tag{2}$$

For the remaining commutator, we find

$$[A_\alpha, A_\beta^\dagger] = \delta_{\alpha\beta} - \Delta_{\alpha\beta},$$

where

$$\Delta_{\alpha\beta} \equiv \sum_{\mu\nu\mu'} \overline{\Phi}_\alpha^{\mu\nu} \Phi_\beta^{\mu'\nu} a_{\mu'}^\dagger a_\mu + \sum_{\mu\nu\nu'} \overline{\Phi}_\alpha^{\mu\nu} \Phi_\beta^{\mu\nu'} b_{\nu'}^\dagger b_\nu.$$

For the matrices Φ_α, we require the normalization condition

$$\text{Tr}(\Phi_\alpha \Phi_\beta^\dagger) = \delta_{\alpha\beta}. \tag{3}$$

The entity $\Delta_{\alpha\beta}$ embodies a deviation from the pure bosonic commutation relation. Note that a pure boson (when $\Delta_{\alpha\beta} = 0$) is not a particular case of a quasi-boson, because $\Delta_{\alpha\beta} = 0$ would require $\Phi_\alpha = 0$, which would yield the invalidity of the very composite structure (1).

Note that, unlike the realization of quasi-bosonic operators using the quonic variant of the deformed oscillator algebra, as it was done in [2], the considered copies of a deformed oscillator will be completely independent in our treatment below. That is, we will assume the validity of (2) and $[A_\alpha, A_\beta^\dagger] = 0$ for $\alpha \neq \beta$.

3. Can the Arik–Coon Type Deformed Oscillators Model the Quasi-Bosons?

Here, we will model the quasi-bosons by the (independent) system of q-deformed bosons of the Arik–Coon type. The latter obey

$$[\mathcal{A}_\alpha, \mathcal{A}_\beta^\dagger] = \delta_{\alpha\beta} + (q^{\delta_{\alpha\beta}} - 1)\mathcal{A}_\beta^\dagger \mathcal{A}_\alpha, \tag{4}$$

where the independence of modes is guaranteed by $\delta_{\alpha\beta}$.

The quasi-bosonic number operator $\mathcal{N}_\alpha$ is defined as

$$\mathcal{N}_\alpha = \log_q \left(1 + (q-1)\mathcal{A}_\alpha^\dagger \mathcal{A}_\alpha\right),$$

which is the inversion of $\mathcal{A}^\dagger \mathcal{A} = \frac{q^\mathcal{N} - 1}{q - 1}$, see [4].

We recall that the Arik–Coon model system involves, in addition, the relations

These relations imply that the operator $\mathcal{A}_\alpha^\dagger$ is the raising operator for deformed bosons (correspondingly, $\mathcal{A}_\alpha$ – lowering operator).

Our goal is to find such coefficients $\Phi_\alpha^{\mu\nu}$ that realization (1) is in agreement with (4), i.e. that relation (4) is valid on the appropriate space of states. If the ground state $|O\rangle$ for quasi-bosons is defined as

$$A_\alpha|O\rangle = a_\mu|O\rangle = b_\nu|O\rangle = 0,$$

then the respective space of states is nothing but the linear span $\{|O\rangle, A_{\gamma_1}^\dagger|O\rangle, A_{\gamma_2}^\dagger A_{\gamma_1}^\dagger|O\rangle, \ldots\}$. Rewrite the commutation relations (4) as

$$F_{\alpha\beta} \equiv \Delta_{\alpha\beta} + (q^{\delta_{\alpha\beta}} - 1)A_\beta^\dagger A_\alpha = 0.$$

Then the validity of commutation relations on the indicated linear span reduces to nullifying each of the states $|O\rangle$, $A^\dagger_{\gamma_1}|O\rangle$, $A^\dagger_{\gamma_2}A^\dagger_{\gamma_1}|O\rangle$, ... by the operator $F_{\alpha\beta}$.

Obviously, for the ground state, we have

$$F_{\alpha\beta}|O\rangle = 0.$$

Notice that

$$F_{\alpha\beta}A^\dagger_{\gamma_1}|O\rangle = 0 \qquad \Leftrightarrow \qquad [F_{\alpha\beta}, A^\dagger_{\gamma_1}]|O\rangle = 0,$$
$$F_{\alpha\beta}A^\dagger_{\gamma_1}A^\dagger_{\gamma_2}|O\rangle = 0 \qquad \Leftrightarrow \qquad [[F_{\alpha\beta}, A^\dagger_{\gamma_1}], A^\dagger_{\gamma_2}]|O\rangle = 0.$$

The equality $[F_{\alpha\beta}, A^\dagger_{\gamma_1}]|O\rangle = 0$ can be rewritten in the form of a relation on matrices Φ_α. Using this, the commutator reduces to

$$[F_{\alpha\beta}, A^\dagger_{\gamma_1}] = (1 - q^{\delta_{\alpha\beta}})A^\dagger_\beta \left[F_{\alpha\gamma_1} + (1 - q^{\delta_{\alpha\gamma_1}})A^\dagger_{\gamma_1} A_\alpha \right].$$

Calculate the double commutator:

$$[[F_{\alpha\beta}, A^\dagger_{\gamma_1}], A^\dagger_{\gamma_2}] = (1 - q^{\delta_{\alpha\beta}})A^\dagger_\beta[F_{\alpha\gamma_1}, A^\dagger_{\gamma_2}] +$$
$$+ (1 - q^{\delta_{\alpha\beta}})(1 - q^{\delta_{\alpha\gamma_1}})A^\dagger_\beta A^\dagger_{\gamma_1}[A_\alpha, A^\dagger_{\gamma_2}].$$

From this equation, for the relation $[[F_{\alpha\beta}, A^\dagger_{\gamma_1}], A^\dagger_{\gamma_2}]|O\rangle = 0$ to hold, we infer

$$(1 - q^{\delta_{\alpha\beta}})(1 - q^{\delta_{\alpha\gamma_1}})\delta_{\alpha\gamma_2} A^\dagger_\beta A^\dagger_{\gamma_1}|O\rangle = 0.$$

Thus, we come to the contradiction: at $\alpha = \beta = \gamma_1 = \gamma_2$ and $q \neq 1$, it follows that

$$(A^\dagger_\alpha)^2|O\rangle = 0,$$

the paradoxical fact – nilpotency of "bosonic" operators.

Hence, the Arik–Coon type deformation, see (4), leads to the inconsistency and so is inappropriate for a realization of quasi-bosons. The situation changes, however, for other deformations, as will be seen below.

4. Quasi-Bosons vs Deformation of General Form

In what follows, we study the independent quasi-bosons system realized by deformed oscillators *without indication of a particular model of deformation*. In this section, we obtain the necessary conditions for such realization in terms of the structure function and matrices Φ_α.

Let ϕ be the structure function of deformation. The quasi-boson number operator is introduced as

$$N_\alpha \stackrel{\text{def}}{=} \phi^{-1}(A^\dagger_\alpha A_\alpha).$$

Note that this definition is not unique. Another equivalent definition could be given, e.g., $N_\alpha \stackrel{\text{def}}{=} \phi^{-1}(A_\alpha A^\dagger_\alpha) - 1$. Below, we need the notion of weak equality denoted by the symbol $\cong$. Namely, if G is some operator function, then its weak equality (to zero) means

$$G(A, A^\dagger, N; ...) \cong 0 \stackrel{\text{def}}{\Leftrightarrow} G(...)A^\dagger_{\gamma_m}... A^\dagger_{\gamma_1}|O\rangle = 0 \qquad (5)$$

for $m = 0, 1, 2, ...$.

4.1. Derivation of necessary conditions

We require the validity of the following weak equalities for commutators:

$$[N_\alpha, A^\dagger_\alpha] \cong A^\dagger_\alpha, \quad [N_\alpha, A_\alpha] \cong -A_\alpha,$$
$$[A_\alpha, A^\dagger_\beta] \cong 0 \quad \text{if } \alpha \neq \beta, \qquad\qquad (6)$$
$$[A_\alpha, A^\dagger_\alpha] \cong \phi(N_\alpha + 1) - \phi(N_\alpha).$$

We also emphasize that, whatever is the definition of N_α, the following implications must be true:

$$\phi(N_\alpha) \cong A^\dagger_\alpha A_\alpha \qquad \Rightarrow \quad \phi(0) = 0,$$
$$\phi(N_\alpha + 1) \cong A_\alpha A^\dagger_\alpha \qquad \Rightarrow \quad \phi(1) = 1.$$

From the second relation in (6), the equality

$$\sum_{\mu'\nu'} \left(\Phi^{\mu\nu'}_\beta \overline{\Phi}^{\mu'\nu'}_\alpha \Phi^{\mu'\nu}_\gamma + \Phi^{\mu\nu'}_\gamma \overline{\Phi}^{\mu'\nu'}_\alpha \Phi^{\mu'\nu}_\beta \right) = 0, \quad \alpha \neq \beta, \quad (7)$$

does follow, which can be rewritten in the matrix form

$$\Phi_\beta \Phi^\dagger_\alpha \Phi_\gamma + \Phi_\gamma \Phi^\dagger_\alpha \Phi_\beta = 0, \quad \alpha \neq \beta. \qquad (8)$$

Since $A^\dagger_\alpha A_\alpha \cong \phi(N_\alpha)$ and $A_\alpha A^\dagger_\alpha \cong \phi(N_\alpha + 1)$, we have

$$[A^\dagger_\alpha A_\alpha, A_\alpha A^\dagger_\alpha] \cong 0 \qquad \text{and} \qquad [\Delta_{\alpha\alpha}, N_\alpha] \cong 0.$$

The first of these relations can be rewritten as

$$[A^\dagger_\alpha A_\alpha, \Delta_{\alpha\alpha}] \cong 0.$$

After calculations, this commutator takes the form

$$[A^\dagger_\alpha A_\alpha, \Delta_{\alpha\alpha}] = 2A^\dagger_\alpha \sum_{\mu\nu} \left(\Phi_\alpha \Phi^\dagger_\alpha \Phi_\alpha \right)^\dagger_{\nu\mu} b_\nu a_\mu -$$
$$- 2 \sum_{\mu'\nu'} \left(\Phi_\alpha \Phi^\dagger_\alpha \Phi_\alpha \right)_{\mu'\nu'} a^\dagger_{\mu'} b^\dagger_{\nu'} A_\alpha \cong 0. \qquad (9)$$

We denote the matrix in parentheses by $\Psi_\alpha \equiv \Phi_\alpha \Phi^\dagger_\alpha \Phi_\alpha$. For the weak equality in (9) to be valid, it is necessary

that the following commutator with the creation operator give zero on the vacuum state:

$$\left[(\overline{\Psi}_\alpha^{\mu\nu}\Phi_\alpha^{\mu'\nu'} - \overline{\Phi}_\alpha^{\mu\nu}\Psi_\alpha^{\mu'\nu'})a_{\mu'}^\dagger b_{\nu'}^\dagger, b_\nu a_\mu, \Phi_\alpha^{\lambda\rho}a_\lambda^\dagger b_\rho^\dagger\right]|O\rangle =$$

$$= \left(\Phi_\alpha^{\mu'\nu'}\cdot\mathrm{Tr}(\Psi_\alpha^\dagger\Phi_\alpha) - \Psi_\alpha^{\mu'\nu'}\right)a_{\mu'}^\dagger b_{\nu'}^\dagger|O\rangle = 0. \tag{10}$$

This leads to the requirement

$$\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha = \mathrm{Tr}(\Phi_\alpha^\dagger\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha)\cdot\Phi_\alpha, \tag{11}$$

which is also the sufficient one. Then we come to two requirements, (8) and (11), for the matrices Φ_α.

4.2. Relating Φ_α to the structure function $\phi(n)$

Now let us derive the relations that involve the structure function ϕ. For the commutator $[A_\alpha, A_\alpha^\dagger]$, we have

$$[A_\alpha, A_\alpha^\dagger] = 1 - \Delta_{\alpha\alpha} \cong \phi(N_\alpha + 1) - \phi(N_\alpha).$$

From the latter,

$$F_{\alpha\alpha} \equiv \Delta_{\alpha\alpha} - 1 + \phi(N_\alpha + 1) - \phi(N_\alpha) \cong 0.$$

If the conditions (see the first line in (6))

$$[N_\alpha, A_\alpha^\dagger] \cong A_\alpha^\dagger, \quad [N_\alpha, A_\alpha] \cong -A_\alpha \tag{12}$$

do hold (this means that, for these conditions, a verification is needed, see Sec. 4.3 below), then

$$\phi(N_\alpha)A_\alpha^\dagger \cong A_\alpha^\dagger\phi(N_\alpha + 1) \quad \Rightarrow$$

$$\Rightarrow \quad [\phi(N_\alpha), A_\alpha^\dagger] \cong A_\alpha^\dagger\big(\phi(N_\alpha + 1) - \phi(N_\alpha)\big).$$

That leads to the relation

$$[F_{\alpha\alpha}, A_\alpha^\dagger] \cong 2(\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha)^{\mu\nu}a_\mu^\dagger b_\nu^\dagger +$$

$$+ A_\alpha^\dagger\big(\phi(N_\alpha + 2) - 2\phi(N_\alpha + 1) + \phi(N_\alpha)\big). \tag{13}$$

From the requirement that this commutator vanish on the vacuum state, we obtain (note that $\phi(0) = 0$):

$$\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha = \left(\phi(1) - \frac{1}{2}\phi(2)\right)\Phi_\alpha = \frac{f}{2}\Phi_\alpha,$$

where the (deformation) parameter f is introduced:

$$\frac{f}{2} \equiv \phi(1) - \frac{1}{2}\phi(2) = \mathrm{Tr}(\Phi_\alpha^\dagger\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha) \text{ for all } \alpha.$$

Then equality (13) takes the form

$$[F_{\alpha\alpha}, A_\alpha^\dagger] \cong f\cdot A_\alpha^\dagger + A_\alpha^\dagger\big(\phi(N_\alpha+2)-2\phi(N_\alpha+1)+\phi(N_\alpha)\big).$$

By induction, the equality for the n-th commutator (C_n^k denote binomial coefficients) can be proven:

$$[...[F_{\alpha\alpha}, A_\alpha^\dagger]...A_\alpha^\dagger] \cong (A_\alpha^\dagger)^n\left\{\sum_{k=0}^{n+1}(-1)^{n+1-k}C_{n+1}^k\phi(N_\alpha+k)\right\}.$$

Using the requirement that the n-th commutator vanish on the vacuum state, we derive the recurrence relation

$$\phi(n+1) = \sum_{k=0}^{n}(-1)^{n-k}C_{n+1}^k\phi(k), \quad n \geq 2. \tag{14}$$

Thus, all the values $\phi(n)$ for $n \geq 3$ are determined unambiguously by the values $\phi(1)$ and $\phi(2)$ which depend, in general, on some set of deformation parameters.

It can easily be shown that the non-deformed structure function $\phi(n) \equiv n$ satisfies Eq. (14). Similarly, one proves the following natural "initial" conditions:

$$\phi(1) \to 1, \quad \phi(2) \to 2 \quad \Rightarrow \quad \forall k > 2 \quad \phi(k) \to k,$$

when all the deformation parameters tend to their non-deformed values.

Taking into account the equality [22]

$$\sum_{k=0}^{n}C_n^k k^m(-1)^{n-k} = \begin{cases}0, & m < n, \\ n!, & m = n,\end{cases}$$

we see that the only independent solutions of the recurrence relation (14) are n and n^2, as well as their linear combination

$$\phi(n) = \left(1 + \frac{f}{2}\right)n - \frac{f}{2}n^2. \tag{15}$$

This formula satisfies both the initial conditions and the recurrence relations (14). In view of the uniqueness of the solution under fixed initial conditions, formula (15) gives the general solution of relation (14).

4.3. Verification of relations (12)

As we have mentioned above, it remains to satisfy relations (12). Note that the second of them stems by conjugation from the first one,

$$[N_\alpha, A_\alpha^\dagger] \cong A_\alpha^\dagger. \tag{16}$$

In view of the *independence* of different modes, see (7), it is enough to set $\gamma_1 = \gamma_2 = \ldots = \alpha$ on the r.h.s. of (5).

Let us denote, by I_n, the operators

$$I_0 = N \equiv \phi^{-1}(A_\alpha^\dagger A_\alpha),$$

$$I_{n+1} = [I_n, A_\alpha^\dagger] = [...[N_\alpha, A_\alpha^\dagger]...A_\alpha^\dagger]. \tag{17}$$

In terms of these operators, Eq. (16) is written as

$$I_1|O\rangle = A_\alpha^\dagger|O\rangle, \quad I_n|O\rangle = 0, \quad n > 1. \tag{18}$$

Introduce the notation

$$\varepsilon_\alpha \equiv 1 - \Delta_{\alpha\alpha} = [A_\alpha, A_\alpha^\dagger].$$

Using the auxiliary relations

$$[\Delta_{\alpha\alpha}, A_\alpha^\dagger] = f A_\alpha^\dagger, \quad [\Delta_{\alpha\alpha}, A_\alpha] = -\overline{f} A_\alpha,$$

$$[\varepsilon_\alpha, A_\alpha^\dagger] = -f A_\alpha^\dagger, \quad [\Delta_{\alpha\alpha}, N_\alpha] \cong 0, \quad \Delta_{\alpha\alpha} = \Delta_{\alpha\alpha}^\dagger,$$

we come to the equalities

$$\left[(A_\alpha^\dagger A)^n, A_\alpha^\dagger\right] = A_\alpha^\dagger \left[(A_\alpha^\dagger A_\alpha + \varepsilon_\alpha)^n - (A_\alpha^\dagger A_\alpha)^n\right], \tag{19}$$

$$\left[\varepsilon_\alpha^n, A_\alpha^\dagger\right] = A_\alpha^\dagger[(-f + \varepsilon_\alpha)^n - \varepsilon_\alpha^n]. \tag{20}$$

From these equalities, we derive the appropriate expression for the n-fold commutator (17) ($\alpha_n = n(n-1)/2$):

$$I_n = (A_\alpha^\dagger)^n \phi^{-1}(A_\alpha^\dagger A_\alpha + n\varepsilon_\alpha - \alpha_n f) - \sum_{k=0}^{n-1} C_n^k (A_\alpha^\dagger)^{n-k} I_k.$$

Finally, conditions (18) can be cast in the form

$$A_\alpha^\dagger \phi^{-1}(A_\alpha^\dagger A_\alpha + \varepsilon_\alpha)|O\rangle = A_\alpha^\dagger|O\rangle,$$

$$(A_\alpha^\dagger)^n \phi^{-1}(A_\alpha^\dagger A + n\varepsilon_\alpha - \alpha_n f)|O\rangle = n(A_\alpha^\dagger)^n|O\rangle, \ n > 1.$$

To satisfy the first of these equalities, we require that

$$\phi^{-1}(1) = 1 \quad \Rightarrow \quad \phi(1) = 1.$$

Likewise, the second equality to be valid requires: $\phi^{-1}\left(n - \frac{n(n-1)}{2}f\right) = n$. This gives us the "bonus" in the form of expression (15) for the structure function.

Remark. Using the obtained results, it is easy to derive the recurrence relation for the structure function,

$$\phi(n+1) = \frac{2(n+1)}{n}\phi(n) - \frac{n+1}{n-1}\phi(n-1),$$

and, for the Hamiltonian $H = \frac{1}{2}(\phi(n+1) + \phi(n))$, the recurrence relation for its eigenvalues (the energies):

$$E_{n+1} = \frac{4n^2 + 4n - 4}{2n^2 - 1}E_n - \frac{2n^2 + 4n + 1}{2n^2 - 1}E_{n-1}.$$

The latter has typical form of the so-called quasi-Fibonacci relation [13]. The general class of deformed oscillators with polynomial structure functions $\phi(N)$ (these are quasi-Fibonacci as well) was studied in [15].

5. Admissible Matrices Φ_α

It remains to find the admissible matrices Φ_α. These should satisfy (3) and the equations

$$\Phi_\alpha \Phi_\alpha^\dagger \Phi_\alpha = (f/2)\Phi_\alpha, \tag{21}$$

$$\Phi_\beta \Phi_\alpha^\dagger \Phi_\gamma + \Phi_\gamma \Phi_\alpha^\dagger \Phi_\beta = 0, \alpha \neq \beta. \tag{22}$$

Let us assume $f \neq 0$. If $\det \Phi_\alpha \neq 0$ for some-α, Eq. (21) yields

$$\Phi_\alpha \Phi_\alpha^\dagger = \frac{f}{2}\mathbf{1}.$$

From Eq. (22) at $\gamma = \alpha$, we obtain

$$\Phi_\beta = 0, \quad \forall \beta \neq \alpha.$$

Then it follows that only one value of α is possible, for which $\det \Phi_\alpha \neq 0$. In that case, Φ_α is an arbitrary unitary matrix. All the rest $\Phi_\beta = 0$, $\beta \neq \alpha$. That gives the partial non-degenerate solution of the system. All other solutions will be degenerate for all α.

Now what concerns the case of degenerate solutions. Using some facts from linear algebra (the Fredholm theorem, *etc.*), we come to the following implication:

$$\mathrm{Tr}(\Phi_\alpha \Phi_\alpha^\dagger) = 1 \quad \Rightarrow \quad \mathrm{rank}\,(\Phi_\alpha) = 2/f \equiv m \text{ for all } \alpha.$$

So, the deformation parameter f has a discrete range of values (if the set of indices μ, ν is finite or enumerable):

$$f = \frac{2}{m} \quad \Rightarrow \quad \phi(n) = \left(1 + \frac{1}{m}\right)n - \frac{1}{m}n^2. \tag{23}$$

The set of the solutions depends on the relation between d and $k \cdot m$, where k denotes the number of independent copies (modes) of deformed bosons, and d is the smallest dimension among the dimensions of matrices Φ_α. If $d \cdot m > d$, the set of solutions is empty. If $k \cdot m \leq d$, then there exist such unitary matrices U_1 and U_2 that the following matrix product is block-diagonal:

$$U_1^\dagger \Phi_\alpha U_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{\Phi}_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the $m \times m$ matrix $\tilde{\Phi}_\alpha$ obeys the equation

$$\tilde{\Phi}_\alpha \tilde{\Phi}_\alpha^\dagger = \frac{f}{2}\mathbf{1}.$$

Its general solution can be given through the unitary matrix

$$\tilde{\Phi}_\alpha = \sqrt{f/2}\, U_\alpha(m).$$

Thus, the general solution of Eqs. (21) and (22) is

$$\Phi_\alpha = U_1 \mathrm{diag}\left\{0, \sqrt{\frac{f}{2}} U_\alpha(m), 0\right\} U_2^\dagger, \tag{24}$$

where, for every matrix Φ_α, the block $\sqrt{\frac{f}{2}} U_\alpha(m)$ in (24) is at the α-th place and has zero intersection with the corresponding block of any other matrix Φ_β for $\beta \neq \alpha$.

6. Concluding Remarks

Here, we make a kind of resume, also pointing out some further directions. For the system of completely independent quasi-bosons, their representation in terms of deformed bosons of the AC type fails. Nevertheless, the desired realization is possible with some other structure function ϕ of the form (15), i.e. with the structure function which is *quadratic in the number operator* and contains one parameter of deformation. The additional very important necessary and sufficient conditions on the matrices Φ_α involved in construction (1) of quasi-bosons, for such representation to be consistent, are derived. They can be completely solved which results in the general solution (24).

Although we used pure fermions as constituents, the analysis shows that the parameter of deformation giving the quasi-boson's realization (see (23)) is linked with a discrete characteristic m (the rank) of the matrix Φ_α.

As the further nearest goals, it is interesting to study more complicated situations. First of all, it is natural to extend the construction of quasi-bosons, formed from two particles, to the case of the constituents that are not fermions but a (particular or general) deformation of fermions. Some results already obtained in this direction will be published separately. Another path of the extension is the treatment of quasi-independent quasi-bosons, in which case one should start with a proper definition of the "physical" subspace of quasi-bosonic states.

Note added in proof. Recently, the above results have been extended to the case of quasi-bosons composed of two q-deformed fermions [23]. In addition, using the results of the present work, the relation between the entanglement in composite (fermion+fermion) bosons realized by deformed bosons and the parameter of deformation, is established [24].

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КВАЗІБОЗОНИ, СКЛАДЕНІ З ДВОХ ФЕРМІОНІВ, ТА ДЕФОРМОВАНІ ОСЦИЛЯТОРИ

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Р е з ю м е

Поняття квазібозонів чи складених бозонів має широкий спектр фізичних застосувань (мезони, екситони тощо). Відомо, що навіть у випадку квазібозонів, складених із двох звичайних ферміонів, їх оператори народження і знищення задовольняють нестандартні комутаційні співвідношення. Природно спробувати реалізувати квазібозонні оператори відповідно операторами народження і знищення деформованих (нелінійних) осциляторів, адже останні становлять добре вивчену область сучасної квантової фізики. У статті доведено, що такі деформовані осцилятори, які реалізують квазібозони, справді існують. Виведено необхідні і достатні умови для реалізації. Також доведено єдиність сім'ї можливих деформацій.